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Complex surfaces which are fibre bundles

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Abstract

We characterize complex surfaces admitting holomorphic submersions to complex curves and quotients of such surfaces by free actions of finite groups in terms of their Euler characteristics and fundamental groups. © 2000 Elsevier Science B.V. All rights reserved.

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It is an easy consequence of the classification of surfaces that a minimal compact complex surface S is ruled over a curve C of genus $\geqslant 2$ if and only if $\pi_1(S) \cong \pi_1(C)$ and $\chi(S) = 2\chi(C)$. (See Chapter VI of [2].) We shall give a similar characterization of the complex surfaces which admit holomorphic submersions to complex curves of genus $\geqslant 2$, and more generally of quotients of such surfaces by free actions of finite groups. However we shall use the classification only to handle the cases of non-Kähler surfaces.

In order to avoid possible ambiguities, we shall use the term "n-manifold" to mean C^∞ manifold of (real) dimension n, and reserve the words "curve" and "surface" for complex manifolds of complex dimension 1 and 2, respectively. We shall moreover use the term "PD_2^+-group" for the fundamental group of an aspherical closed orientable 2-manifold.

Theorem 1. *Let S be a compact complex surface. Then S has a finite covering space which admits a holomorphic submersion onto a complex curve, with base and fibre of genus $\geqslant 2$, if and only if $\pi = \pi_1(S)$ has normal subgroups $K < G$ such that K and G/K are PD_2^+-groups, $[\pi : G] < \infty$ and $[\pi : G]\chi(S) = \chi(K)\chi(G/K) > 0$.*

Proof. The conditions are clearly necessary. Suppose that they hold. Then S is aspherical, by Theorem IV.1 of [5]. In particular, π is torsion free and $\pi_2(S) = 0$, so S is minimal.

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After enlarging K if necessary we may assume that π/K has no nontrivial finite normal subgroup. Let S_G be the finite covering space corresponding to G. Then $\beta_1(S_G) \geqslant 4$. If $\beta_1(S_G)$ were odd then S_G would be minimal properly elliptic, by the classification of surfaces. But then either $\chi(S) = 0$ or S_G would have a singular fibre and the projection of S_G to the base curve would induce an isomorphism on fundamental groups [3]. Hence $\beta_1(S_G)$ is even and so S_G and S are Kähler (see Theorem 4.3 of [13]). Since π/K is not virtually Z^2 it is isomorphic to a discrete group of isometries of the upper half plane $\mathbb{H}^2$ and $\beta_1^{(2)}(\pi/K) \neq 0$. Hence there is a properly discontinuous holomorphic action of π/K on $\mathbb{H}^2$ and a π/K-equivariant holomorphic map from the covering space S_K to $\mathbb{H}^2$, with connected fibres, by Theorems 4.1 and 4.2 of [1]. Let B and B_G be the complex curves $\mathbb{H}^2/(\pi/K)$ and $\mathbb{H}^2/(G/K)$, respectively, and let $h : S \to B$ and $h_G : S_G \to B_G$ be the induced maps. The quotient map from $\mathbb{H}^2$ to B_G is a covering projection, since G/K is torsion free, and so $\pi_1(h_G)$ is an epimorphism with kernel K.

The map h is a submersion away from the preimage of a finite subset $D \subset B$. Let F be the general fibre and F_d the fibre over $d \in D$. Fix small disjoint discs $\Delta_d \subset B$ about each point of D, and let $B^* = B - \bigcup_{d\in D} \Delta_d$, $S^* = h^{-1}(B^*)$ and $S_d = h^{-1}(\Delta_d)$. Since $h|_{S^*}$ is a submersion $\pi_1(S^*)$ is an extension of $\pi_1(B^*)$ by $\pi_1(F)$. The inclusion of ∂S_d into $S_d - F_d$ is a homotopy equivalence. Since F_d has real codimension 2 in S_d the inclusion of $S_d - F_d$ into S_d is 2-connected. Hence $\pi_1(\partial S_d)$ maps onto $\pi_1(S_d)$.

Let $m_d = [\pi_1(F_d) \colon \operatorname{Im}(\pi_1(F))]$. After blowing up S at singular points of F_d we may assume that it has only normal crossings. We may then pull $h|_{S_d}$ back over a suitable branched covering of Δ_d to obtain a singular fibre $\widetilde{F}_d$ with no multiple components and only normal crossing singularities. In that case $\widetilde{F}_d$ is obtained from F by shrinking vanishing cycles, and so $\pi_1(F)$ maps onto $\pi_1(\widetilde{F}_d)$. Since blowing up a point on a curve does not change the fundamental group it follows from Section 9 of Chapter III of [2] that in general m_d is finite.

We may regard B as an orbifold with cone singularities of order m_d at $d \in D$. By the Van Kampen theorem (applied to the space S and the orbifold B) the image of $\pi_1(F)$ in π is a normal subgroup and h induces an isomorphism from $\pi/\pi_1(F)$ to $\pi_1^{orb}(B)$. Therefore the kernel of the canonical map from $\pi_1^{orb}(B)$ to $\pi_1(B)$ is isomorphic to $K/\operatorname{Im}(\pi_1(F))$. But this is a finitely generated normal subgroup of infinite index in $\pi_1^{orb}(B)$, and so must be trivial. Hence $\pi_1(F)$ maps onto K, and so $\chi(F) \leqslant \chi(K)$.

Let D_G be the preimage of D in B_G. The general fibre of h_G is again F. Let F_{G_d} denote the fibre over $d \in D_G$. Then $\chi(S_G) = \chi(F)\chi(B) + \sum_{d\in D_G}(\chi(F_{G_d}) - \chi(F))$ and $\chi(F_{G_d}) \geqslant \chi(F)$, by Proposition III.11.4 of [2]. Moreover $\chi(F_{G_d}) > \chi(F)$ unless $\chi(F_{G_d}) = \chi(F) = 0$, by Remark III.11.5 of [2]. Since $\chi(B_G) = \chi(G/K) < 0$, $\chi(S_G) = \chi(K)\chi(G/K)$ and $\chi(F) \leqslant \chi(K)$ it follows that $\chi(F) = \chi(K) < 0$ and $\chi(F_{G_d}) = \chi(F)$ for all $d \in D_G$. Therefore $F_{G_d} \cong F$ for all $d \in D_G$ and so h_G is a holomorphic submersion. $\square$

We may construct examples of such surfaces as follows. Let $n > 1$ and C_1 and C_2 be two curves such that $\mathbb{Z}/n\mathbb{Z}$ acts freely on C_1 and with isolated fixed points on C_2. Then the quotient S of $C_1 \times C_2$ under the induced action is a complex surface and the projection from $C_1 \times C_2$ to C_2 induces a surjective holomorphic mapping from S to $C_2/(\mathbb{Z}/n\mathbb{Z})$ with critical values corresponding to the fixed points.

Corollary. *The surface S admits such a holomorphic submersion onto a complex curve if and only if π has a normal subgroup K such that K and π/K are PD_2^+-groups and $\chi(S) = \chi(K)\chi(\pi/K) > 0$.*

The referee has provided the following example to show that in general the Euler characteristic condition is necessary, even if we assume S minimal. Let C be a curve of genus > 2, and let $Y = C \times C \times P^1$. Choose a degree d embedding of Y into a projective space, and let X be a general hyperplane section of Y. Then $\pi_1(X) \cong \pi_1(C) \times \pi_1(C)$, by the Lefshetz hyperplane theorem, but $\chi(X) \to \infty$ as $d \to \infty$.

Similar arguments may be used to characterize surfaces ruled over curves of genus $\geqslant 2$ and to show that a Kähler surface S is a minimal properly elliptic surface with no singular fibres if and only if $\chi(S) = 0$ and $\pi = \pi_1(S)$ has a normal subgroup $A \cong \mathbb{Z}^2$ such that π/A is virtually torsion free and indicable, but is not virtually Abelian. Moreover, if S is not a ruled surface then it is a complex torus, a hyperelliptic surface, an Inoue surface, a Kodaira surface or a minimal elliptic surface if and only if $\chi(S) = 0$ and $\pi_1(S)$ has a normal subgroup A which is poly-$\mathbb{Z}$ and not cyclic, and such that π/A is infinite and virtually torsion free indicable. (See Chapter X, §5 of [5].)

We may combine Theorem 1 with some observations deriving from the classification of surfaces for our second result.

Theorem 2. *Let S be a complex surface such that $\pi = \pi_1(S) \neq 1$. If S is homotopy equivalent to a C^∞ 4-manifold E which fibres over a compact orientable 2-manifold then S is diffeomorphic to E.*

Proof. Let B and F be the base and fibre of the bundle, respectively. Suppose first that $\chi(F) = 2$. Then $\chi(B) \leqslant 0$, for otherwise S would be simply-connected. Hence $\pi_2(S)$ is generated by an embedded S^2 with self-intersection 0, and so S is minimal. Therefore S is ruled over a curve diffeomorphic to B, by the classification of surfaces.

Suppose next that $\chi(B) = 2$. If $\chi(F) = 0$ and $\pi \ncong \mathbb{Z}^2$ then $\pi \cong \mathbb{Z} \oplus (\mathbb{Z}/n\mathbb{Z})$ for some $n > 0$. Then S is a Hopf surface and so is determined up to diffeomorphism by its homotopy type, by Theorem 12 of [7]. If $\chi(F) = 0$ and $\pi \cong \mathbb{Z}^2$ or if $\chi(F) < 0$ then S is homotopy equivalent to $S^2 \times F$, so $\chi(S) < 0$, $w_1(S) = w_2(S) = 0$ and S is ruled over a curve diffeomorphic to F. Hence E and S are diffeomorphic to $S^2 \times F$.

In the remaining cases E and F are both aspherical. If $\chi(F) = 0$ and $\chi(B) \leqslant 0$ then $\chi(S) = 0$ and π has one end. Therefore S is a complex torus, a hyperelliptic surface, an Inoue surface, a Kodaira surface or a minimal properly elliptic surface. (This uses Bogomolov's theorem on class VII_0 surfaces [10].) The Inoue surfaces are mapping tori of self-diffeomorphisms of $S^1 \times S^1 \times S^1$, and their fundamental groups are not extensions of $\mathbb{Z}^2$ by $\mathbb{Z}^2$, so S cannot be an Inoue surface. As the other surfaces are Seifert fibred 4-manifolds E and S are diffeomorphic, by [12].

If $\chi(F) < 0$ and $\chi(B) = 0$ then S is a minimal properly elliptic surface. Let A be the normal subgroup of the general fibre in an elliptic fibration. Then $A \cap \pi_1(F) = 1$ (since $\pi_1(F)$ has no nontrivial Abelian normal subgroup) and so $[\pi : A.\pi_1(F)] < \infty$. Therefore E is finitely covered by a Cartesian product $T \times F$, and so is Seifert fibred. Hence E and S are diffeomorphic, by [12].

The remaining case ($\chi(B) < 0$ and $\chi(F) < 0$) is an immediate consequence of Theorem 1, since such bundles are determined by the corresponding extensions of fundamental groups (see Theorem IV.1 of [5]). □

A simply-connected smooth 4-manifold which fibres over a 2-manifold must be homeomorphic to $CP^1 \times CP^1$ or $CP^2 \sharp \overline{CP^2}$. (See [4], or Chapter IX of [5].) Is there such a surface of general type? (Qin has shown that no surface of general type is diffeomorphic to $CP^1 \times CP^1$ or $CP^2 \sharp \overline{CP^2}$ [9].)

Corollary. *If moreover the base has genus* 0 *or* 1 *or the fibre has genus* 2 *then S is finitely covered by a Cartesian product.*

Proof. A holomorphic submersion with fibre of genus 2 is the projection of a holomorphic fibre bundle and hence S is virtually a product, by [6]. □

Up to deformation there are only finitely many algebraic surfaces with given Euler characteristic > 0 which admit holomorphic submersions onto curves [8]. By the argument of the first part of Theorem 1 this remains true without the hypothesis of algebraicity, for any such complex surface must be Kähler, and Kähler surfaces are deformations of algebraic surfaces (see Theorem 4.3 of [13]). Thus the class of bundles realized by complex surfaces is very restricted. (On the other hand, every closed 4-manifold which fibres over an orientable 2-manifold with fibre an orientable 2-manifold of genus $\geqslant 2$ is symplectic [11].) Which extensions of PD_2^+-groups by PD_2^+-groups are realized by complex surfaces (i.e., not necessarily aspherical)?

Note added in proof

M. Kapovich and D. Kotschick have each independently obtained similar results. Is there a symplectic analogue of Theorem 2?

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